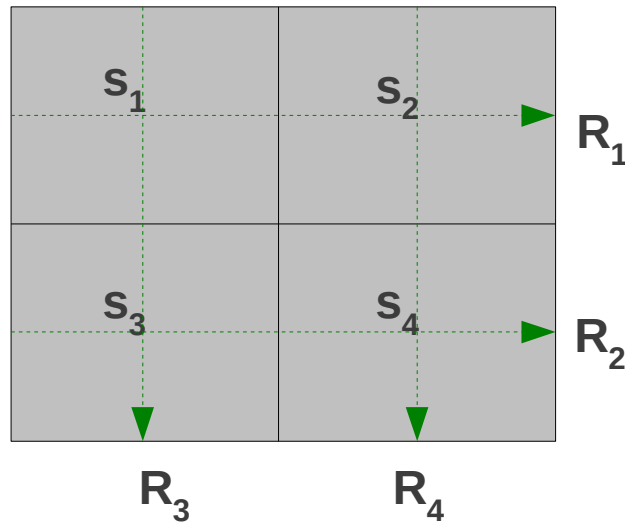# GPH 503 Homework Assignment 6
Straight Ray tomography, Underdetermined and Mixed Determined Problems and Damped Least Squares

Consider the geometry above, where an experiment has been designed to understand the velocity structure of a 2D square object. 4 rays have been shot, R$_1$, R$_2$, R$_3$ and R$_4$ with the ray path geometry above and associated travel times t$_1$, t$_2$, t$_3$ and t$_4$ have been recorded.

The object size is 2 km by 2km and it is divided into 4 1 km by 1 km boxes. Each box has velocity v and slowness s, where $s=1/v$. For example for ray 1 travel time is $t_1=1/v_1+1/v_2$ or $t_1=1s_1+1s_2+0s_3+0s_4$.

t

1. Construct the equations for the travel time $As=b$. Your unknowns s 's are the slownesses in each box (which you will turn to velocity at the end). Your data is the travel time of each ray. Your matrix A consists of the path lengths in each box (in this case they are all 1 km).
2. Since all the rays are along the sides, they traverse 1 km in each box. I wrote the 1st line of the matrix (for R$_1$) and you need to fill out the rest of the rays.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$

3. What is the rank of A? Do elimination to obtain the rank of A. When the columns of A are not independent, we cannot apply $\hat{s}=A(A^T A)^{-1}A^T t$ because $(A^T A)$ is singular and inverse does not exist. Least square solution can only be obtained if columns of A are independent.

For the next questions use the travel time data $t^T = \begin{bmatrix} 0.90 & 0.62 & 0.79 & 0.73 \end{bmatrix}$. I will also give you the answer directly that $\begin{bmatrix} v_1, & v_2, & v_3, & v_4 \end{bmatrix} = \begin{bmatrix} 2.0, & 2.5, & 3.5, & 3 \end{bmatrix}$ are the velocities which were used to obtain those travel times. The next questions will be on two ways to solve this problem. You need to compare your estimates of slowness or velocity to the values above.

4. **Solution to the Underdetermined Problem**
   We somehow need to obtain the slownesses and convert them to velocities. The first way is to remove 1 data point and make the columns of A independent. So we turn the problem into a purely underdetermined problem.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

   Then the solution to purely underdetermined becomes
$$s^{ud} = A^T (AA^T)^{-1} A^T t$$

   Calculate your s^{ud} vector in MATLAB and turn them to velocities using v=1./sud in matlab. Compare your estimated s^{ud} with the data.

5. **Damped Least-Squares Solution to the Mixed Determined Problem**
   When we turn the problem into purely underdetermined problem, we are not using all of the data that we have. The other way to solve it is to use all the data points but add some constraint. One such constraint is to add minimization of the solution length. So in that case the error that we try to minimize is $\Phi(m) = E + \epsilon^2 L = e^T e + \epsilon^2 m^T m$ where m's are the model parameters.

   This leads to following estimation
$$s^{damped} = (A^T A + \epsilon^2 I)^{-1} A^T t$$

   ϵ^2 is something that we have added artificially to make the columns of $A^T A$ independent. This is called regularization, or damping. It is not clear what number this ϵ should be from the beginning. Try several ϵ^2 and compare your estimate of the slownesses to the input values.